RESEARCH STATEMENT

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My research is in combinatorics, a branch of mathematics that focuses on the study of discrete structures. Instances of discrete structures arise in disciplines such as physics, computer science, and other branches of mathematics. Oftentimes, the complexity of discrete structures can make understanding them difficult. As a result, we study combinatorial objects which are easier to understand, but still capture the structure. The most common question we answer about a colletion of objects is "can we enumerate them?", as answering this often reveals much about the object. We can also study **actions** on an object to learn more about that object. Much of my research has been in two distinct combinatorial areas, seeking to answer these questions. I will first outline these areas before going into further depth.

(1) **Toggle Dynamics**

We can start with an object and repeatedly apply an action to an object, we get a sequence of objects. This process terminates if we return to our original object; we will call the sequence of objects an orbit. A typical question we answer is "can we show the orbits exhibit nice properties?" An example application of this is a connection with the six vertex model of statistical physics. Part of the Razumov-Stroganov Correspondence that relates combinatorics and statistical physics can be explained by studying orbits of **posets** under **toggle actions** [13].

Toggles are local actions used to define global actions. Examples include the actions **rowmotion** and **promotion** on order ideals of a poset. The orbits of these actions exhibit interesting properties. J. Propp and T. Roby isolated a phenomenon in which a statistic on a set has the same average value over any orbit as its global average, naming it **homomesy**. In [8], they proved that the cardinality statistic on order ideals of the product of two chains poset under rowmotion and promotion exhibits homomesy. In my research, I built on this to prove an analogous result in the case of the product of three chains where one chain is of length two. In order to prove this result, I generalized the recombination technique from [6] of D. Einstein and J. Propp from two to n dimensions. I also proved a number of corollaries, including a new result on increasing tableaux and a new result on a Type B minuscule poset cross a chain of length two.

I have also been working with Dilks and Striker on toggle dynamics of **increasing labelings**, which generalize results from [3]. There is an important connection between order ideals of posets and increasing tableaux; we generalize this with a connection between increasing labelings and order ideals of a more general class of posets. Additionally, rowmotion previously has only been defined in the setting of a ranked poset cross a chain. We extend this to the product of any two ranked posets.

(2) **Non-attacking Rooks**

An introductory combinatorics question one might ask is "how many ways can you place m rooks on an $n \times n$ chessboard so that no two rooks are attacking each other?" Subsequent research has extended this question to deeper theory, altering the initial question in ways such as replacing rooks with pieces that move in different ways, determining the maximum number of non-attacking pieces one could place on a board, or investigating connections to topics such as generating functions and **permutations**. Several graduate students, my advisor, and I found an interesting problem by exploring a particular three-person chessboard. We then generalized this board to a wider class of boards, obtained by chaining square boards together. In [7], we counted maximum placements of non-attacking rooks on these boards and established connections to permutations and alternating sign matrices.

1. Toggle Dynamics

1.1. **Posets and homomesy.** A **partially ordered set**, or poset, is a general object and can be used to represent integers under divisibility, subsets of a given set under inclusion, roots of Lie groups and reflection groups, among others. A special subset of a poset is an **order ideal**; if an element is in an order ideal, all elements less than or equal to that element must also be in the order ideal. Another way to say this is that order ideals are closed downward; the notion relates to ring ideals of algebra. We define an action on order ideals called a **toggle**; we toggle an element of a poset in the following way. If an element is not in an order ideal and adding it would result in an order ideal, we do so. If an element is in an order ideal and removing it would result in an order ideal, we do so. Otherwise, we leave the element alone. By composing toggles, we obtain interesting actions on order ideals.

One such action is **rowmotion**. When performing rowmotion on a poset, we toggle each element in the poset in the order of top to bottom. Rowmotion has generated significant interest in recent years [2, 3, 6, 8, 10, 14, 16]. This research is useful in the field of quantum computing, as a particular class of posets captures quantum structure [16]. Order ideals exhibit another interesting action called **promotion**. For posets in two dimensions, promotion can be described as toggling each element in the poset in the order of left to right. Striker and Williams showed the following intimate connection between rowmotion and promotion.

Theorem 1.1 ([14])**.** *For any ranked poset, there is an equivariant bijection between the order ideals under promotion and under rowmotion.*

In other words, order ideals under rowmotion and promotion have the same orbit structure. Striker and Williams found that in many cases, it was easier to determine the orbit sizes of promotion compared to rowmotion. The reason for this is that in many cases, promotion is in equivariant bijection with rotation on another object. As a result, when studying one of these actions, it can be helpful to instead study the other.

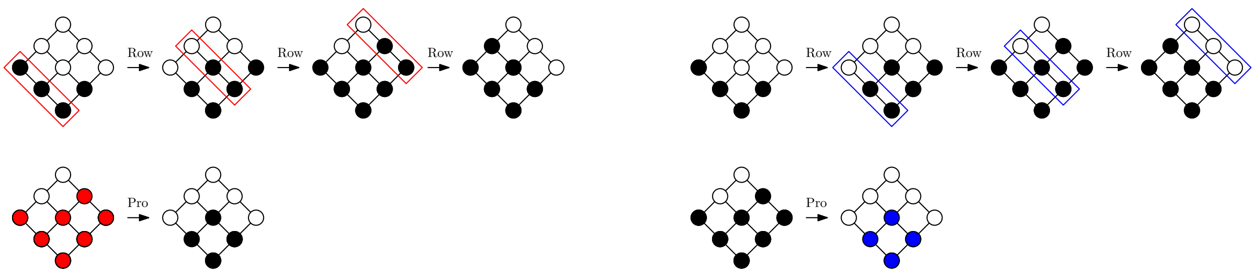
Following this work, Dilks, Pechenik, and Striker generalized promotion to higher dimensions [3]. Instead of sweeping through a two dimensional poset from left to right to determine toggle order, we sweep through an n-dimensional poset with an affine hyperplane to determine toggle order; we get a different promotion for each affine hyperplane. Additionally, we note that one of these promotions is rowmotion.

We can also ask ourselves what it means for our orbits to be "nice". One property that indicates an orbit is nice is **homomesy**. An object, an action, and a statistic exhibit homomesy if the average of the statistic over every orbit is the same as the global average of the statistic. Homomesy is a widespread phenomenon, with examples found in actions on tableaux [1, 8], actions on binary strings [9], rotations on permutation matrices [9], toggles on noncrossing partitions [5], Suter's action on Young diagrams [8] (with proof due to D. Einstein), linear maps acting on vector spaces [8], a phase-shift action on simple harmonic motion [8], and others. Propp and Roby discovered homomesy in a product of chains poset $[a] \times [b]$ for both promotion and rowmotion with statistic the cardinality of the order ideal.

Theorem 1.2 ([8])**.** *Promotion on the order ideals of* $[a] \times [b]$ *with the cardinality statistic is homomesic with average value* $ab/2$.

Theorem 1.3 ([8])**.** *Rowmotion on the order ideals of* $[a] \times [b]$ *with the cardinality statistic is homomesic with average value* $ab/2$.

Striker and Williams were able to relate the orbit structure of promotion and rowmotion; we might wonder if there is a connection between the homomesy results of rowmotion and promotion. Einstein and Propp answered this in the affirmative for $[a] \times [b]$ using a technique they called **recombination** [6]. By taking particular layers of the posets from an orbit under rowmotion, we can put these together to form an orbit under promotion. See Figure 4.



(A) From an orbit of rowmotion, we used the boxed layers to form a new order ideal, denoted here in red.

(B) From the same orbit of rowmotion, we use the boxed layers to form a new order ideal, denoted here in blue.

FIGURE 1. Performing promotion on the red order ideal results in the blue order ideal.

1.2. **Results.** With the generalization of promotion to higher dimensions [3], natural questions to ask are whether the homomesy and recombination results generalize to higher dimensions. In my research, I have answered the question about homomesy in the affirmative if the product of chains is of the form $[2] \times [a] \times [b]$. More specifically, using the order ideal cardinality statistic and any of the three dimensional promotion actions, $J([2] \times [a] \times [b])$ exhibits homomesy with average ab.

Theorem 1.4 ([15])**.** *For any* $v = (\pm 1, \pm 1, \pm 1)$, Pro_v *on the order ideals of* $[2] \times [a] \times [b]$ *with the cardinality statistic is homomesic with average value* ab.

Additionally, we can move the coordinate of the chain with length two, getting a similar result for $[a] \times [2] \times [b]$ and $[a] \times [b] \times [2]$. Using code in SageMath, I also determined that the result does not hold in general in three dimensions, nor does the result hold in higher dimensions even when using just chains of length two. In other words, my result in Theorem 1.4 is the best result we could attain.

The method I used to prove Theorem 1.4 is a generalized recombination result for not just three dimensions, but n dimensions. The result says that if the vector defining our affine hyperplane differs in one coordinate, we can take particular layers of our poset from an orbit under one of our promotions and put them together to form an orbit under the other promotion.

Theorem 1.5 ([15])**.** *Let I be an order ideal of $[a_1]\times\cdots\times[a_n]$. Suppose we have $v = (v_1, v_2, \ldots, v_n)$ where $v_j = \pm 1$, $u = (u_1, u_2, \ldots, u_n)$ where $u_j = \pm 1$, $v_\gamma = 1$, $u_\gamma = -1$, and u and v are the same in all other coordinates. Then $\mathrm{Pro}_u(\Delta_v^\gamma I)$=$\Delta_v^\gamma(\mathrm{Pro}_v(I))$.*

Proving Theorem 1.4 also required a bijection introduced in [3] to objects called **increasing tableaux** under an action called **K-promotion**. More specifically, I used a homomesy result from [1] on increasing tableaux translated to the poset setting. After proving Theorem 1.4, I use the same bijection to obtain a new homomesy result on increasing tableaux.

Corollary 1.6 ([15])**.** *Let λ be an $a \times b$ rectangle and let σ_λ be the statistic of summing the entries in the boxes of λ. Then $(\mathrm{Inc}^{a+b+1}(\lambda), K\text{-Pro}, \sigma_\lambda)$ is c-mesic with $c = ab + \frac{ab(a+b)}{2} = \frac{ab(2+a+b)}{2}$.*

Additionally, I generalized my recombination result from the product of chains setting to all ranked posets, a much more general class of posets. See Figures 2 and 3 for an example.

Theorem 1.7 ([15])**.** *Let P be a poset with n-dimensional lattice embedding π and $I \in J(P)$. Suppose we have $v = (v_1, v_2, \ldots, v_n)$ where $v_j = \pm 1$, $u = (u_1, u_2, \ldots, u_n)$ where $u_j = \pm 1$, $v^* = (v_1, \ldots, v_{\gamma-1}, v_{\gamma+1}, \ldots, v_n)$, $u^* = (u_1, \ldots, u_{\gamma-1}, u_{\gamma+1}, \ldots, u_n)$ and γ such that $v_\gamma = 1$, $u_\gamma = -1$, and $v^* = u^*$. Then $\mathrm{Pro}_{\pi,u}(\pi^{-1}(\Delta_v^\gamma(\pi(I)))) = \pi^{-1}(\Delta_v^\gamma(\pi(\mathrm{Pro}_{\pi,v}(I))))$.*

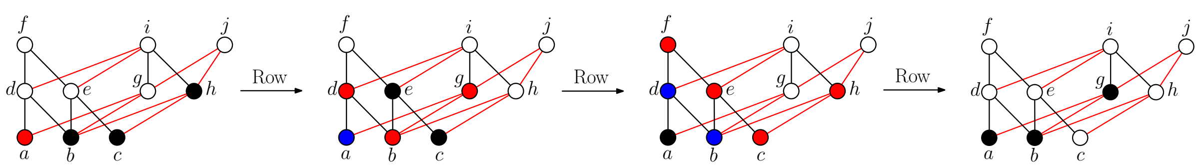


Figure 2. We use the red layers and blue layers from the partial orbit to form two new order ideals.

Applying this theorem results in an easy corollary yielding homomesy on a Type B minuscule poset cross a chain of length two.

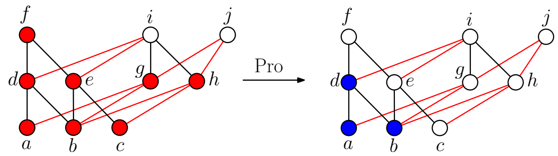


FIGURE 3. Applying promotion to the red order ideal gives us the blue order ideal.

Corollary 1.8 ([15]). *Let f be the cardinality statistic and B_n denote the type B minuscule poset of size n. For $v = \{\pm 1, \pm 1, \pm 1\}$, the triple $(J(B_n \times [2]), \mathrm{Pro}_v, f)$ is c-mesic with $c = \frac{n^2+n}{2}$.*

1.3. **Increasing labelings.** In the above section, I refered to a bijection from [3] between the product of three chains poset and increasing tableaux. Working with Dilks and Striker, we generalized increasing tableaux to objects called **increasing labelings** [4]. More specifically, if P is a poset, a function $f : P \to \mathbb{Z}$ is an increasing labeling if $x <_P y$ implies that $f(x) < f(y)$. Increasing labelings have similar properties as increasing tableaux, but can take a much more general shape. Our goal was to generalize the bijection between increasing tableaux and the product of chains poset to increasing labelings and a larger class of posets.

1.4. **Results.** By constructing the right poset, $\Gamma_1(P, R)$, we were able to establish a bijection between increasing labelings and order ideals of this poset.

Theorem 1.9 ([4]). *Order ideals of $\Gamma_1(P, R)$ are in bijection with increasing labels on P with restriction R.*

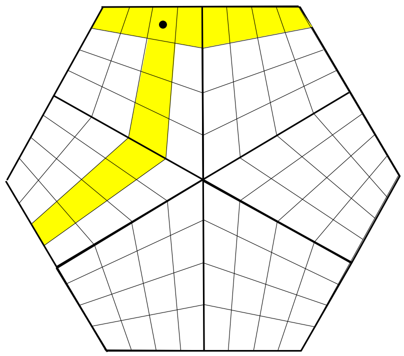
The motivating bijection is an equivariant bijection between the product of chains poset under rowmotion and increasing tableaux under K-promotion. Our next goal was to find an analogous action on increasing tableaux to establish an equivariant bijection to $\Gamma_1(P, R)$ under rowmotion. By generalizing the definition of K-promotion, we were able to define such an action, which we also called promotion. From here, we obtained the desired result.

Theorem 1.10 ([4]). *Let P be a poset and $q \in \mathbb{N}$. There is an equivariant bijection between $\mathrm{Inc}^q(P)$ under promotion and order ideals in $\Gamma_1(P, q)$ under rowmotion.*

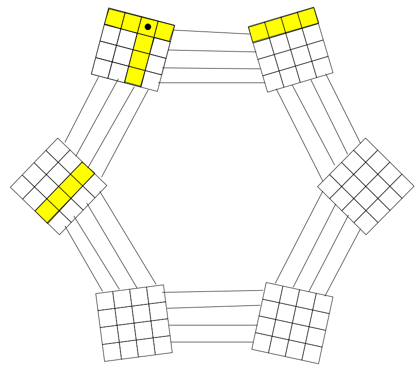
2. Non-attacking rooks

2.1. **Results.** The three-person chessboard that intialized this research is shown in Figure 4a. By reorganizing the boards, as seen in Figures 4b and 5, we obtained an object consisting of 6 chained

4×4 square boards. We generalize this to k chained $n \times n$ square boards. We also consider the difference between connecting the first and last board (circular case) or having them unconnected (linear case).



(A) A three-person chessboard; the dot represents a rook and the highlighted cells are the cells the rook is attacking.



(B) We pull the board apart and represent connecting ranks and files with an edge.

FIGURE 4

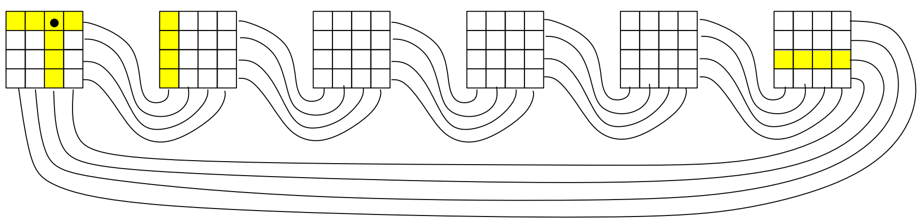


FIGURE 5. By reorganizing boards, we can generalize to a larger class of boards.

Our first main result states how many ways we can place m non-attacking rooks on k chained $n \times n$ board, for both the circular and linear case.

Theorem 2.1 ([7]). *The number of ways to place m non-attacking rooks on board $B \in \{B^{-}_{n,k}, B^{\circ}_{n,k}\}$ is* $\displaystyle\sum_{(a_1,\ldots,a_k)\in\mathfrak{C}_m(B)} \prod_{i=1}^{k} \binom{n-a_{i-1}}{a_i} (n)_{a_i}$, *where a_0 is defined as follows:* $a_0 = \begin{cases} 0 & \text{if } B = B^{-}_{n,k} \\ a_k & \text{if } B = B^{\circ}_{n,k}. \end{cases}$

From here, our enumeration inqueries focused on maximum placements of non-attacking rooks. In other words, for a particular board, what is the largest number of non-attacking rooks we can place and how many ways can we place them? We determined the maximum number of non-attacking rooks for each type of board, then used Theorem 2.1 to enumerate the number of placements.

Theorem 2.2 ([7]). *The number of maximum rook placements on $B^{-}_{n,k}$ is given by:*

- *Case k even:* $(n!)^{\frac{k}{2}} \sum_{0\le j_1\le\ldots\le j_{\frac{k}{2}}\le n} \prod_{\ell=1}^{\frac{k}{2}} \binom{n-j_{\ell-1}}{n-j_\ell}\binom{n}{j_\ell}.$
- *Case k odd:* $(n!)^{\frac{k+1}{2}}$,

Theorem 2.3 ([7])**.** *The number of maximum rook placements on $B^{\circ}_{n,k}$ is given by:*

- *Case k even:* $(n!)^{\frac{k}{2}} \sum_{j=0}^{n} \binom{n}{j}^{\frac{k}{2}}$,
- *Case k odd, n even:* $\left((n)_{\frac{n}{2}}\right)^k$,
- *Case k odd, n odd:* $k\left((n)_{\lceil\frac{n}{2}\rceil}\right)^{\lfloor\frac{k}{2}\rfloor}\left((n)_{\lfloor\frac{n}{2}\rfloor}\right)^{\lceil\frac{k}{2}\rceil}.$

A square board of size $n\times n$ with n non-attacking rooks is in bijection with $n\times n$ **permutation matrices**, and as a result, permutations of n. Because our objects are square boards chained together, it was natural to define the notion of a **chained permutation**. We generalized several concepts related to permutations to their chained permutation analogues, including **one-line notation** and **perfect matchings**. Furthermore, $n\times n$ **alternating sign matrices** generalize $n\times n$ permutation matrices; we were able to define a chained analogue of alternating sign matrices as well.

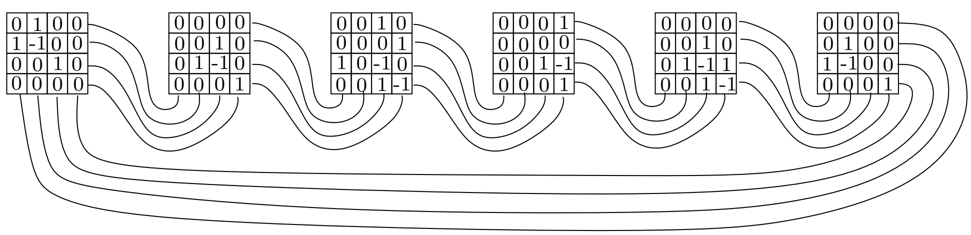


FIGURE 6. A 4×4 chained alternating sign matrix where $k=6$.

For some cases, we were able to enumerate the number of distinct **chained alternating sign matrices**.

Corollary 2.4 ([7])**.** *For k odd,* $|ASM^{-}_{n,k}| = \left(\prod_{k=0}^{n-1}\frac{(3k+1)!}{(n+k)!}\right)^{\frac{k+1}{2}}.$

Corollary 2.5 ([7])**.** $|ASM^{\circ}_{n,4}| = \prod_{k=0}^{2n-1}\frac{(3k+1)!}{(2n+k)!}.$

Corollary 2.6 ([7])**.** $|ASM^{\circ}_{2m,1}| = \left(\prod_{k=0}^{m-1}\frac{(3k+1)!}{(m+k)!}\right)^3 \prod_{i=1}^{m}\left(\frac{3i-1}{3i-2}\prod_{j=i}^{m}\frac{m+i+j-1}{2i+j-1}\right).$

Although we could not find a general enumeration formula, I wrote code to generate all chained alternating sign matrices for small n and k. Additionally, we defined chained analogues of objects in bijection with alternating sign matrices: **monotone triangles**, **square ice configurations**, and **fully-packed loops**.

3. Future Work

All previously stated toggle action results have applied strictly to finite posets. I am currently generalizing rowmotion and promotion to ranked infinite posets. Much of the intuition from the finite case is lost, as we are no longer guaranteed to obtain finite orbits. Because of this, I am looking for connections to other fields such as dynamical systems and algebra for both intuition and to see if these interesting actions would be useful answering questions in other fields. Monomial ideals are a promising avenue, as order ideals and monomial ideals are closely connected. Furthermore, the action of rowmotion was originally defined in terms of generators, and the generators of monomial ideals are heavily studied by algebraists.

As mentioned above, I have frequently used the open source software SageMath [12] to perform calculations for my research. In addition to writing a significant amount of code for myself, I've also contributed code that has been included in SageMath. I plan to write more code to investigate future projects and contribute my recent homomesy code to SageMath.

One strength of combinatorics is that there are a wide variety of topics that would make great projects for undergraduates. Much of my previous research can be explained to an undergraduate student without a tremendous amount of background theory. Additionally, there are several natural directions to proceed from my results, such as searching for homomesy on different posets or using different statistics. Another great aspect of combinatorics research is the availability of SageMath to compute examples. If a student has a background in programming, writing code in SageMath would be an excellent way for them to contribute to a research project; if a student doesn't have a strong background in programming, writing small pieces of code would be a good way to improve their coding skills. I have put some of these ideas into practice, as I am currently advising an undergraduate student on his senior capstone project at North Dakota State.

Although I mentioned my main research above, I have also done combinatorial research in the field of Ramsey theorey [11]. More specifically, for my Master's thesis at South Dakota State, I studied colorings on the integers. To give a broad summary: given two colors and an inequality, I found the length required to guarantee there were integers all in the same color that satisfied the inequality. I have enjoyed all three areas of research, toggle dynamics, rook placements, and Ramsey theory. Moreover, I would love to continue research in any three of these areas as part of undergraduate projects.

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